INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 41, Northern Autumn 2019 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Let the number of prime factors in the prime factorisation of an integer $n > 1$ be called the *complexity* of n. For example, the complexity of numbers $4 = 2 \times 2$ and $6 = 2 \times 3$ is equal to 2 for both numbers. For which integers n do all integers m strictly between n and $2n$, i.e. satisfying $n + 1 \leq m \leq 2n - 1$, have complexity

(a) not greater than the complexity of n? (2 points)

(b) less than the complexity of n? (2 points)

2. Let ABC and $A_1B_1C_1$ be two acute triangles such that points B_1 and C_1 lie on the side BC and A_1 lies inside triangle ABC. Let S and S_1 be the areas of triangles ABC and $A_1B_1C_1$ respectively. Prove that

$$\frac{S}{AB + AC} > \frac{S_1}{A_1B_1 + A_1C_1}.$$

(7 points)

3. There are 100 visually identical coins of three types: gold, silver and bronze, with at least one coin of each type. Each gold coin weighs 3 grams, each silver coin weighs 2 grams and each bronze coin weighs 1 gram. How can one determine for sure the type of each coin by making no more than 101 weighings on a set of balance scales with no weights? (7 points)

4. Let ABC be a triangle with circumcentre O. Two perpendiculars OP and OQ are dropped from O onto the internal and external bisectors of $\angle B$. Prove that the line PQ bisects the line segment that connects the midpoints of CB and AB. (7 points)

5. Let a pair (m, n) of distinct positive integers be called *nice* if mn and $(m+1)(n+1)$ are perfect squares. Prove that for every positive integer m there exists at least one n with $n > m$ such that the pair (m, n) is nice. (8 points)

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6. Petya initially has several \$100 notes and no other money. He begins to buy books, each book costing a whole number of dollars, and he receives change in \$1 coins. If Petya buys an expensive book (\$100 or higher), he uses only \$100 notes, tendering the smallest number of \$100 notes required. If he is buying a cheap book (less than \$100), he uses his \$1 coins if he has enough, otherwise he uses a \$100 note. Petya finally runs out of \$100 notes, and at that time he had spent exactly half of his money. Is it possible that Petya spent at least \$5000 on books? (8 points)

7. Petya has a wooden square stamp divided into a grid of small squares. He covers 102 small squares of the stamp with black ink. Then he presses the stamp 100 times on a sheet of white paper so that each time only the blackened 102 squares are imprinted on the paper. Is it possible that after doing the imprint on the paper there is a 101×101 square grid such that all its 1×1 small squares except one corner square are black? (10 points)